

SPECTRAL ANALYSIS OF ONE-DIMENSIONAL HIGH-CONTRAST ELLIPTIC PROBLEMS WITH PERIODIC COEFFICIENTS

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Abstract. We study the behaviour of the spectrum of a family of one-dimensional operators with periodic high-contrast coefficients as the period goes to zero, which may represent *e.g.* the elastic or electromagnetic response of a two-component composite medium. Compared to the standard operators with moderate contrast, they exhibit a number of new effects due to the underlying non-uniform ellipticity of the family. The effective behaviour of such media in the vanishing period limit also differs notably from that of multi-dimensional models investigated thus far by other authors, due to the fact that neither component of the composite forms a connected set. We then discuss a modified problem, where the equation coefficient is set to a positive constant on an interval that is independent of the period. Formal asymptotic analysis and numerical tests with finite elements suggest the existence of localised eigenfunctions (“defect modes”), whose eigenvalues situated in the gaps of the limit spectrum for the unperturbed problem.

Key words. Elliptic differential equations, homogenisation, spectrum

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1. Introduction.

1.1. The general context for the problem in hand. The description of the effective behaviour of high-contrast composites (“high-contrast homogenisation”) has been of particular interest in the analysis and applied communities over the last decade. The analytical part of the related literature starts with the work [17], which developed in detail some earlier ideas of [1] concerning the use of “two-scale convergence” for the analysis of the limit behaviour of the boundary-value problem

$$-\operatorname{div}(\mathcal{A}^\varepsilon(x/\varepsilon)\nabla u) = f, \quad f \in L^2(\Omega), \quad u \in H_0^1(\Omega), \quad \mathcal{A}^\varepsilon = \varepsilon^2\chi_0 I + \chi_1 I, \quad \varepsilon > 0,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, and χ_0, χ_1 are the indicator functions of $[0, 1)^n$ -periodic sets in \mathbb{R}^n such that $\chi_0 + \chi_1 = 1$.

Several contributions to the high-contrast homogenisation followed: in the linear and non-linear, scalar and vector contexts, with various sets of assumptions about the underlying geometry of the composite. With applications mainly in solid mechanics and electromagnetism, high-contrast media have served as a theoretical ground for a number of effects observed in physics experiments, in particular those related to photonic band-gap materials and cloaking metamaterials ([14]). The range of techniques developed in these contexts and their applications continue their rapid expansion, and the present paper is one contribution aimed at addressing some aspects that have thus far been left out of the scope of the related research.

More specifically, we approach the question of the analysis of the spectral behaviour of high-contrast composites in the case when the component represented by the function χ_1 (the “matrix” of the composite) is disconnected in \mathbb{R}^n . Clearly, this is always the case in one dimension ($n = 1$), which is the situation we study in the present article.

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1.2. Problem setup. We consider solutions u to the following family of elliptic problems on an interval $(a, b) \subset \mathbb{R}$:

$$A^\varepsilon u - \lambda u = f, \quad f \in L^2(a, b), \quad \varepsilon > 0, \quad \lambda \in \mathbb{C}, \quad (1.1)$$

where the operators A^ε are given by the closed bilinear form

$$(A^\varepsilon u, v) = \int_a^b p(x/\varepsilon) (\varepsilon^2 \chi_0(x/\varepsilon) + \chi_1(x/\varepsilon)) u'(x) \overline{v'(x)} dx, \quad u, v \in \mathfrak{H}. \quad (1.2)$$

Here $p = p(y) > 0$ is a 1-periodic function in \mathbb{R} such that $p, p^{-1} \in L^\infty(0, 1)$, the functions χ_0 and χ_1 are the indicator functions of 1-periodic open sets F_0 and F_1 such that $\overline{F_0} \cup \overline{F_1} = \mathbb{R}$, and \mathfrak{H} denotes a closed linear subspace of $H^1(a, b)$ that contains $C_0^\infty(a, b)$. We make no assumptions regarding boundedness of the interval (a, b) , in particular it may coincide with the whole space \mathbb{R} .

In applied contexts the problem (1.1) corresponds to, *e.g.*, the study of wave propagation in a layered 2D or 3D composite structure where $f = 0$, $\lambda > 0$. In what follows we study the spectrum S^ε of the problem (1.1), *i.e.* the set of values of λ for which $A^\varepsilon - \lambda I$ does not have a bounded inverse in $L^2(a, b)$. Throughout the article we employ the notation $\sigma(A)$ for the spectrum of an operator A , and the notation Q for the “unit cell” $[0, 1]$ whenever we describe the behaviour with respect to the “physical” variables x, y . We continue writing $[0, 1]$ for the “Floquet-Bloch dual” cell when we refer to the domain of the quasimomentum θ .

1.3. Our strategy for the analysis of (1.1). It has been well understood in the existing literature on the subject (see [2], [17], [19]), that in the analysis of convergence of spectra of families of differential operators with periodic rapidly oscillating coefficient, one has to deal with two distinct issues: the lower semicontinuity of the spectra in the sense of Hausdorff convergence of sets, and the possibility of spectral pollution, the lack of which is often referred to as “spectral completeness”. The former issue, which in the wider spectral analytic context has been looked at from a more general perspective (see *e.g.* [4]), is usually dealt with by proving first a variant of strong resolvent convergence. In the case of periodic operators involving multiple scales, one typically makes use of the so-called “two-scale convergence” (see *e.g.* [13], [1], [17]). In the present paper we follow this general approach in proving the related lower semicontinuity statements both for the whole-space problem and for the problem in a bounded interval. It should be pointed out that this first part of the analysis of spectral convergence is not completely independent from the subsequent study of spectral completeness: unless some assumptions are made concerning the geometry of the periodic composite in question (see *e.g.* [17]), one may not get the best possible “lower bound” for the limit spectrum. It has been noticed that, in order to capture the behaviour with respect to all Bloch components in the limit as $\varepsilon \rightarrow 0$, it is preferable to use an advanced, “multi-cell” version, of the standard two-scale convergence; see *e.g.* [2], [5, Chapter 5], where this more refined approach is adopted. It is a version of this last, more detailed, procedure that we adopt in the present article.

In the proof of spectral completeness, a natural strategy seems to try and analyse the relative strength of different Bloch components in a given (convergent) sequence of eigenfunctions. This idea has been elaborated in [2] in the specific context of “high-frequency” homogenisation with the use of what the authors refer to as the “Bloch measures”. A combination of a compactness argument in the related space

of measures and a special “slow-variable modulation” construction then yields the simultaneous convergence of the given sequence to a limit eigenfunction and of the associated eigenvalues. In the present work we suggest an alternative approach (see Section 3) to the convergence of eigenfunctions, which we believe is closer in spirit to the idea of “spectral compactness”, *i.e.* compactness of eigenfunctions in a norm-preserving topology. Our approach is based on the idea that once one has control of the behaviour of eigenfunctions in the orthogonal complement to the space spanned by the limit eigenfunctions, one can immediately pass to the limit, as $\varepsilon \rightarrow 0$, in the weak formulation of the original family of eigenvalue problems. This idea allows us to cover the analysis of spectral convergence for a wide range of operator families, including those considered by [2], [17], [16], [5, Chapter 4].

The key element in our analysis, which allows us to implement the above idea is Proposition 3.2 below (see Section 3.1), or equivalently Proposition 3.3. These statements establish a uniform version of the Poincaré-type inequality between the projection of a given function onto the “poorly behaving” subspace and the L^2 -norm of its derivative on the part of the domain where solutions of the eigenvalue problem can be shown to be *a priori* small as $\varepsilon \rightarrow 0$. Different versions of the same idea have appeared in a number of other contexts, serving a similar purpose of “compensating” somehow the apparent loss of compactness in the problem, for example, in the form of Korn inequality in elasticity (see *e.g.* [6], and also [18] for its multiscale versions), in the form of the so-called “energy method” in classical homogenisation (see [12]), and, more recently, in the form of a “generalised Weyl decomposition” for problems with degeneracies (see [10]). For nonlinear variants of the same idea, the reader may be referred to the “geometric rigidity” (see [7]) and “ \mathcal{A} -quasiconvexity” (see [8]).

For an easier introduction to the problem, in what follows we start with the analysis of the problem (1.1) in the whole-space case, $(a, b) = \mathbb{R}$, see Section 2. While a version of the compactness argument developed in the bounded-interval setting (see Section 3) applies here as well (once complemented by a suitable Weyl-sequence argument), we present a different argument, based on some ideas of [5, Chapter 5], where the spectral analysis is carried out in a more challenging setting of the Maxwell system.

Throughout the article we assume for simplicity that the restriction of χ_0 to the periodicity cell $[0, 1)$ is the indicator function of an open interval (α, β) , which we also denote by Q_0 . We use the notation Q_1 for the interior of the complement of Q_0 to the interval $(0, 1)$.

2. Limit analysis for the whole space. In this section we consider the case $(a, b) = \mathbb{R}$. One well-known procedure for calculating S^ε is the Floquet-Bloch decomposition ([3]) following the rescaling $y = x/\varepsilon$. Then, for $\theta \in (0, 1]$ the sequence of eigenvalues $\lambda = \lambda(\theta)$ corresponding to θ -quasiperiodic solutions to the Floquet-Bloch problem on the interval $(0, 1)$ associated to the differential expression $(p(\varepsilon^2\chi_0 + \chi_1)u)'$ is obtained by solving the dispersion equation

$$\begin{aligned} \frac{1}{2} \left(\frac{1}{\varepsilon} + \varepsilon \right) \sin \left(\varepsilon \sqrt{\lambda} (\alpha - \beta + 1) \right) \sin \left(\sqrt{\lambda} (\alpha - \beta) \right) \\ + \cos \left(\varepsilon \sqrt{\lambda} (\alpha - \beta + 1) \right) \cos \left(\sqrt{\lambda} (\alpha - \beta) \right) = \cos(2\pi\theta). \end{aligned}$$

Passing to the limit in the above equation as $\varepsilon \rightarrow 0$ yields

$$\frac{1}{2} (\alpha - \beta + 1) \sqrt{\lambda} \sin \left(\sqrt{\lambda} (\alpha - \beta) \right) + \cos \left(\sqrt{\lambda} (\alpha - \beta) \right) = \cos(2\pi\theta). \quad (2.1)$$

By varying θ as indicated we obtain (for $\alpha = 1/4$, $\beta = 3/4$) the set shown in Fig. 2.1.

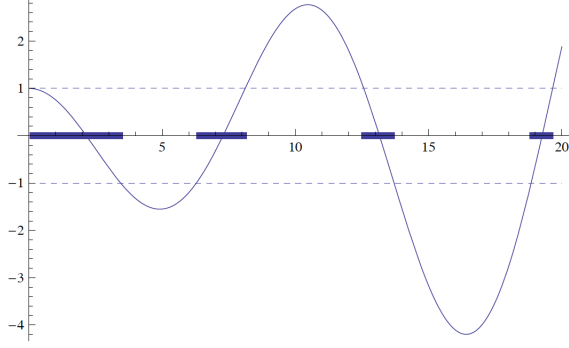


FIG. 2.1. The square root of the limit Bloch spectrum. The oscillating solid line is the graph of the function $f(t) = \cos(t/2) - t \sin(t/2)/4$, where t represents $\sqrt{\lambda}$ in the formula (2.1) with $\alpha = 1/4$, $\beta = 3/4$. The square root of the spectrum is the union of the intervals indicated by bold lines.

Our first result is the following theorem.

THEOREM 2.1. *Let $(a, b) = \mathbb{R}$. Then the set $\lim_{\varepsilon \rightarrow 0} S^\varepsilon$ is given by the union of solution sets for the equation (2.1) for all $\theta \in [0, 1]$.*

For $\theta \in [0, 1)$, we denote by $H_\theta^1(Q)$ the space of functions $u \in H^1(Q)$ that are θ -quasiperiodic, i.e. such that $v(y) = \exp(2\pi i \theta y)u(y)$, $y \in Q$, for some¹ $u \in H_\#^1(Q)$. We also denote

$$V(\theta) := \{v \in H_\theta^1(Q) : p(y)v'(y) = 0 \text{ for } y \in Q_1\}. \quad (2.2)$$

Consider an operator $A(\theta)$ such that

$$(A(\theta)u, \varphi) = \int_{Q_0} p(y)u'(y)\overline{\varphi'(y)}dy \quad \forall \varphi \in V(\theta), \quad u \in \text{dom}(A(\theta)) \subset V(\theta),$$

defined on the maximal possible domain $\text{dom}(A(\theta))$. Henceforth (\cdot, \cdot) denotes the usual inner product in $L^2(Q)$. By a standard argument (see e.g. [11]) such an operator exists, is unique and, under the adopted conditions on the coefficient p , is self-adjoint and has compact inverse (except for the case $\theta = 0$, when it has compact inverse as an operator on $V(\theta) \ominus \mathbb{C}$). Therefore the spectrum $\sigma(A(\theta))$ is discrete and unbounded, i.e. it consists of eigenvalues $0 \leq \lambda_1(\theta) \leq \lambda_2(\theta) \leq \dots$, of finite multiplicity with eigenfunctions $v^k(\theta) = v^k(\theta, y)$. The eigenfunctions corresponding to different eigenvalues are automatically orthogonal in $L^2(Q)$. We also carry out the orthogonalisation process on those eigenfunctions that correspond to the same eigenvalue, and normalise each eigenfunction so that $\|v^k(\theta)\|_{L^2(Q)} = 1$, for all $\theta \in [0, 1)$, $k \in \mathbb{N}$.

Our aim is to show that the limit set $\lim_{\varepsilon \rightarrow 0} S^\varepsilon$ coincides with the union of the spectra of the operators $A(\theta)$, $\theta \in [0, 1)$, which, in turn, are described by the "dispersion relation" (2.1).

¹As the notation $H_0^1(Q)$ is usually reserved for the space of $H^1(Q)$ functions vanishing on the boundary of Q , we denote by $H_\#^1(Q)$ the space $H_\theta^1(Q)$ when $\theta = 0$.

In order to demonstrate first that the latter is included in the former, for each $N \in \mathbb{N}$, we define an “intermediate” operator A_N in $L^2(NQ)$, whose spectrum is contained in $\lim_{\varepsilon \rightarrow 0} S^\varepsilon$ and contains the spectrum of each of the operators $A(\theta)$, $\theta = j/N$, $0 \leq j \leq N-1$. The details of this argument, which relies on a procedure that we refer to as “ NQ -periodic homogenisation”, are given in Appendix A.

An essential component of the proof of the converse inclusion is the following lemma.

LEMMA 2.2. *For any given $\theta \in [0, 1)$ and $\varphi \in V(\theta)$, let $\theta_\varepsilon \in [0, 1)$ be such that $\theta_\varepsilon \rightarrow \theta$ as $\varepsilon \rightarrow 0$. Then there exist $\varphi_\varepsilon \in V(\theta_\varepsilon)$ such that $\varphi_\varepsilon \rightarrow \varphi$ strongly in $H^1(Q)$ as $\varepsilon \rightarrow 0$.*

Proof. For $\theta \in [0, 1)$ the space $V(\theta)$ consists of functions that are θ -quasiperiodic and constant in each connected component of Q_1 , that is for any $\varphi \in V(\theta)$ one has $\varphi(y) = \eta(\theta, y)c + v(y)$, where $c \in \mathbb{C}$, $v \in H_0^1(a, b)$, and

$$\eta(\theta, y) := \begin{cases} 1, & y \in [0, a), \\ (\exp(2\pi i \theta) - 1)(b - a)^{-1}(y - a) + 1, & y \in [a, b], \\ \exp(2\pi i \theta), & y \in (b, 1). \end{cases}$$

For each value of ε we now define φ_ε by the formula $\varphi_\varepsilon(y) = \eta(\theta_\varepsilon, y)c + v(y)$, $y \in Q$. Notice that by construction $\varphi_\varepsilon \in V(\theta_\varepsilon)$ and, since η is uniformly continuous with respect to θ , one has $\varphi_\varepsilon \rightarrow \varphi$ strongly in $H^1(Q)$. \square

We next show that for a sequence $\lambda_\varepsilon \in \sigma(A_\varepsilon)$ such that $\lambda_\varepsilon \rightarrow \lambda$ the inclusion $\lambda \in \sigma(A(\theta))$ holds for some $\theta \in [0, 1)$.

THEOREM 2.3. *Let $\lambda_\varepsilon \in \sigma(A_\varepsilon)$ such that $\lambda_\varepsilon \rightarrow \lambda$. Then there exist $\theta \in [0, 1)$ and $u \in H_\theta^1(Q)$, $u \neq 0$, such that*

$$\int_a^b p(y)u'(y)\overline{\varphi'(y)}dy = \lambda \int_0^1 u(y)\overline{\varphi(y)}dy, \quad \forall \varphi \in V(\theta). \quad (2.3)$$

Proof. Since $\lambda_\varepsilon \in \sigma(A_\varepsilon)$, by the Floquet-Bloch decomposition, there exists $u_\varepsilon \in H_{\theta_\varepsilon}^1(\varepsilon Q)$, $u_\varepsilon \neq 0$, such that

$$\int_{\varepsilon Q} p(x/\varepsilon)(\varepsilon^2 \chi_0(x/\varepsilon) + \chi_1(x/\varepsilon))u'_\varepsilon(x)\overline{\varphi'(x)}dx = \lambda_\varepsilon \int_{\varepsilon Q} u_\varepsilon(x)\overline{\varphi(x)}dx \quad (2.4)$$

for all $\varphi \in H_{\theta_\varepsilon}^1(\varepsilon Q)$. Rescaling the formulation (2.4) with $y = x/\varepsilon$ yields the existence of $u_\varepsilon \in H_{\theta_\varepsilon}^1(Q)$, $\|u_\varepsilon\|_{L^2(Q)} = 1$, such that

$$\varepsilon^{-2} \int_{Q_1} p(y)u'_\varepsilon(y)\overline{\varphi'(y)}dy + \int_{Q_0} p(y)u_\varepsilon(y)\overline{\varphi(y)}dy = \lambda_\varepsilon \int_Q u_\varepsilon(y)\overline{\varphi(y)}dy \quad (2.5)$$

for all $\varphi \in H_{\theta_\varepsilon}^1(Q)$.

The sequence θ_ε is bounded and therefore there exists some $\theta \in [0, 1]$ such that, up to a subsequence which we do not relabel, $\theta_\varepsilon \rightarrow \theta$. Without loss of generality, if $\theta = 1$ we set $\theta = 0$, so that $\theta \in [0, 1)$. By substituting $\varphi = u_\varepsilon$ in (2.5), the sequence u_ε satisfies the bounds

$$\|\chi_1 u'_\varepsilon\|_{L^2(Q)} \leq C\varepsilon, \quad \|\chi_0 u'_\varepsilon\|_{L^2(Q)} \leq C. \quad (2.6)$$

with a constant $C > 0$ independent of ε .

Due to the weak compactness of bounded sets in $H^1(Q)$, the bounds (2.6), along with $\|u_\varepsilon\|_{L^2(Q)} = 1$, imply that, up to extracting a subsequence, u_ε converge weakly in $H^1(Q)$, and therefore strongly in $L^2(Q)$, to some $u_0 \in H^1(Q)$, $\|u_0\|_{L^2(Q)} = 1$. Clearly, for $w_\varepsilon(y) := \exp(-2\pi i \theta_\varepsilon y) u_\varepsilon(y)$ one has $w_\varepsilon \in H^1_\#(Q)$, and the uniform convergence of $\exp(2\pi i \theta_\varepsilon y)$ to $\exp(2\pi i \theta y)$ as $\varepsilon \rightarrow 0$ implies that w_ε converge weakly in $H^1(Q)$ to w_0 given by the formula $w_0(y) = \exp(-2\pi i \theta y) u_0(y)$, so that $u_0 \in H^1_\theta(Q)$. Furthermore, (2.6) implies that $\chi_1 u'_\varepsilon \rightarrow 0$ strongly in $L^2(Q)$, hence $u_0 \in V(\theta)$.

In order to show that u_0 satisfies the limit identity (2.3), for a fixed $\varphi_0 \in V(\theta)$, let $\varphi_\varepsilon \in V(\theta_\varepsilon)$ be given by Lemma 2.2. Substituting φ_ε in (2.5), we obtain

$$\int_a^b p(y) u'_\varepsilon(y) \overline{\varphi'_\varepsilon(y)} dy = \lambda_\varepsilon \int_0^1 u_\varepsilon(y) \overline{\varphi_\varepsilon(y)} dy. \quad (2.7)$$

By virtue of the facts that $\varphi_\varepsilon \rightarrow \varphi_0$ strongly in $H^1(Q)$ and $u_\varepsilon \rightharpoonup u_0$ weakly in $H^1(Q)$, passing to the limit $\varepsilon \rightarrow 0$ in (2.7) immediately implies (2.3). \square

The above “limit spectrum” $\lim_{\varepsilon \rightarrow 0} S^\varepsilon$ is strictly larger than the set obtained by the two-scale analysis of the operator A^ε of the paper [17]. In particular, the spectrum of the homogenised operator obtained in [17] coincides with $\{\lambda_k(0)\}_{k=1}^\infty$, using our notation. Our analysis above shows that the set $\lim_{\varepsilon \rightarrow 0} S^\varepsilon$ has, in fact, a band-gap structure, with infinitely many gaps opening in the interval $[0, \infty)$, as $\varepsilon \rightarrow 0$. This fact suggests possible applications of the above composite structures to the design of optical or acoustic band-gap materials, which we discuss in Section 4. The above effect also raises a mathematical question of the analysis of the limit behaviour of the operators A^ε in the case when (a, b) is a bounded interval, which we study in the next section.

In what follows we assume that $a, b \in \varepsilon F_1$. Our results are also easily carried over to the case when $a, b \in \varepsilon F_0$, if one modifies (1.2) on those connected components of εF_0 that contain a or b , by changing the related coefficient from ε^2 to unity.

3. Spectral behaviour on a bounded interval. It is known that the classical, “moderate-contrast”, analogue of the problem (1.1)–(1.2) leads to limit spectra of different kinds for problems on bounded and unbounded intervals (a, b) : the limit set in the case of the problem in the whole space is purely absolutely continuous while in the case $-\infty < a < b < \infty$ it is purely discrete, *i.e.* it consists of eigenvalues with finite multiplicities, see *e.g.* [3]. A similar situation occurs in multidimensional high-contrast problems where the inclusion $F_0 \cap Q$ has a non-zero distance to the boundary of Q , see [17], where, in addition, some eigenvalues of infinite multiplicity are present.

As we shall see next, this is not the case for the problem (1.1)–(1.2), in particular rescaling $y = x/\varepsilon$ and replacing the form (1.2) with an integral over the whole of \mathbb{R} , leads to higher-order errors in the limit as $\varepsilon \rightarrow 0$, which can be ignored in the leading-order, “homogenised”, description of the operator A^ε .

In this section we employ, for convenience, the following notation: $\Omega := (a, b)$, $\Omega^\varepsilon := \Omega \cap (\varepsilon F_1)$, $\Omega^\varepsilon := \Omega \cap (\varepsilon F_0)$.

3.1. The convergence result. The following theorem holds.

THEOREM 3.1. *Consider an operator A^ε from the class described in Section 1.2, subject to the geometric modification mentioned at the end of Section 2. The set $\lim_{\varepsilon \rightarrow 0} S^\varepsilon$ is given by the union of solutions to the equation (2.1) for all $\theta \in [0, 1)$. In particular, it is independent of the choice of the space \mathfrak{H} in Section 1.2.*

The inclusion of the union of the spectra $\sigma(A(\theta))$ in the set $\lim_{\varepsilon \rightarrow 0} S^\varepsilon$ is proved in the same way as in the case of the whole-space problem, see the proof of Theorem 2.1. In what follows we therefore discuss the converse inclusion (*cf.* Theorem 2.3).

Notice first that for each $\lambda_\varepsilon \in \sigma(A_\varepsilon)$, there exists $u_\varepsilon \in \mathfrak{H}$, $\|u_\varepsilon\|_{L^2(\Omega)} = 1$, such that

$$\int_{\Omega_1^\varepsilon} p\left(\frac{x}{\varepsilon}\right) u'_\varepsilon(x) \overline{\varphi'(x)} dx + \varepsilon^2 \int_{\Omega_0^\varepsilon} p\left(\frac{x}{\varepsilon}\right) u'_\varepsilon(x) \overline{\varphi'(x)} dx = \lambda_\varepsilon \int_{\Omega} u_\varepsilon(x) \overline{\varphi(x)} dx \quad (3.1)$$

for all $\varphi \in \mathfrak{H}$. Setting $\varphi = u_\varepsilon$ in (3.1) yields the estimates (“*a priori* bounds”))

$$\|u'_\varepsilon\|_{L^2(\Omega_1^\varepsilon)} \leq C_B, \quad \varepsilon \|u'_\varepsilon\|_{L^2(\Omega_0^\varepsilon)} \leq C_B, \quad (3.2)$$

where $C_B > 0$ is independent of ε .

For every bounded interval D we denote $D_1 := D \cap F_1$ and introduce the function space (*cf.* (2.2), (4.4), (4.7))

$$V(D) := \{u \in H^1(D) : p(y)u'(y) = 0 \text{ for } y \in D_1\}.$$

The following statement is central to the proof of Theorem 3.1.

PROPOSITION 3.2. *There exists a constant $C_\perp > 0$ such that*

$$\|P_{V(D)^\perp} u\|_{H^1(D)} \leq C_\perp \|u'\|_{L^2(D_1)}$$

for any bounded interval $D \subset \mathbb{R}$ and any $u \in H^1(D)$. Henceforth $V(D)^\perp$ denotes the orthogonal complement of $V(D)$ in the space $H^1(D)$ equipped with the usual inner product, and $P_{V(D)^\perp} u$ denotes the orthogonal projection of the function u onto $V(D)^\perp$.

Consider the norm

$$|||u||| := \left(\left| \int_{Q_1} u(y) dy \right|^2 + \int_Q |u'(y)|^2 dy \right)^{1/2}, \quad u \in H^1(Q), \quad (3.3)$$

which, in view of Lemma 4.3 in Appendix B, is equivalent to the usual H^1 -norm. In the same appendix the above Proposition 3.2 is shown to be equivalent to the following “uniform in θ ” Poincaré-type inequality.

PROPOSITION 3.3. *There exists a constant $\tilde{C} > 0$, which depends on α and β only, such that for any $\theta \in [0, 1)$*

$$|||w|||^2 \leq \tilde{C} \int_{Q_1} |w'(y)|^2 dy, \quad \forall w \in V^\perp(\theta). \quad (3.4)$$

We denote by $V^\perp(\theta)$ the orthogonal complement of $V(\theta)$ defined by (2.2) in the space $H_\theta^1(Q)$ equipped with the inner product in $H^1(Q)$ associated to the norm (3.3).

Proof. We first note some properties of functions that belong to the space $V^\perp(\theta)$, which follow immediately from the characterisation of the space $V(\theta)$ given in the proof of Lemma 2.2.

LEMMA 3.4. *Let $w \in V^\perp(\theta)$, then*

(i) *The equation $w''(y) = 0$ holds for $y \in Q_0$. In particular, the function w is linear on the Q_0 -component of the unit cell: $w(y) = (w(\beta) - w(\alpha))(\beta - \alpha)^{-1}(y - w(\alpha)) + w(\alpha)$ for $y \in Q_0$.*

(ii) For $\theta \neq 0$ one has $(\beta - \alpha)^{-1}(w(\beta) - w(\alpha)) = (1 - \exp(-2\pi i\theta))^{-1}(\alpha + (1 - \beta)\exp(-2\pi i\theta)) \int_{Q_1} w(y)dy$.

(iii) For $\theta = 0$ one has $\int_{Q_1} w(y)dy = 0$.

We now return to the proof of Proposition 3.3. We consider three different cases, depending on the location of the quasimomentum within the Floquet-Bloch cell $[0, 1)$.

Case I: $\theta = 0$. By (3.3) and Lemma 3.4 (i), (iii), we find that

$$|||w|||^2 = \int_{Q_1} |w'(y)|^2 dy + \frac{|w(\beta) - w(\alpha)|^2}{\beta - \alpha}.$$

Since $w(1) = w(0)$, we obtain the estimate

$$\begin{aligned} |||w|||^2 &\leq \int_{Q_1} |w'(y)|^2 dy + \frac{2}{\beta - \alpha} (|w(\beta) - w(1)|^2 + |w(0) - w(\alpha)|^2) \\ &= \int_{Q_1} |w'(y)|^2 dy + \frac{2}{\beta - \alpha} \left(\left| \int_{\beta}^1 w'(y) dy \right|^2 + \left| \int_0^{\alpha} w'(y) dy \right|^2 \right) \\ &\leq \left(1 + \frac{2|Q_1|}{\beta - \alpha} \right) \int_{Q_1} |w'(y)|^2 dy. \end{aligned}$$

Case II: $\theta \in (0, \delta) \cup (1 - \delta, 1)$, where $0 < \delta < 1/2$ is to be chosen appropriately. By (3.3) and Lemma 3.4 (i), (ii), we find that

$$\begin{aligned} |||w|||^2 &= \left| \int_{Q_1} w(y) dy \right|^2 + \int_{Q_1} |w'(y)|^2 dy + \frac{|w(\beta) - w(\alpha)|^2}{\beta - \alpha} \\ &= \left(\frac{1}{\beta - \alpha} + \frac{1}{|d_\theta|^2} \right) |w(\beta) - w(\alpha)|^2 + \int_{Q_1} |w'(y)|^2 dy, \end{aligned}$$

where $d_\theta := (\beta - \alpha)(1 - \exp(-2\pi i\theta))^{-1}(\alpha + (1 - \beta)\exp(-2\pi i\theta))$. From the fact that $w(1) = \exp(2\pi i\theta)w(0)$ we infer

$$\begin{aligned} |w(\beta) - w(\alpha)|^2 &\leq 3 \left(|w(1) - w(\beta)|^2 + |w(\alpha) - w(0)|^2 + |w(1) - w(0)|^2 \right) \\ &\leq 3|Q_1| \int_{Q_1} |w'(y)|^2 dy + 3|\exp(2\pi i\theta) - 1|^2 |w(0)|^2, \end{aligned}$$

and therefore

$$\begin{aligned} |||w|||^2 &\leq \left(3|Q_1| + \frac{1}{\beta - \alpha} + \frac{1}{|d_\theta|^2} \right) \int_{Q_1} |w'(y)|^2 dy \\ &\quad + 3 \left(\frac{1}{\beta - \alpha} + \frac{1}{|d_\theta|^2} \right) |\exp(2\pi i\theta) - 1|^2 |w(0)|^2. \end{aligned}$$

Notice that $|d_\theta|^2 = (\beta - \alpha)^2(2 - 2\cos(2\pi\theta))^{-1}(\alpha^2 + (1 - \beta)^2 + 2\alpha(1 - \beta)\cos(2\pi\theta))$, hence $|d_\theta|$ vanishes at $\theta = 1/2$ for the special case $\alpha = 1 - \beta$. In view of this observation and in order to have a bound on the constant d_θ we require that $\delta < 1/4$. Further, by continuity of the embedding of $H^1(Q)$ in $C(\overline{Q})$, there exists a constant \hat{c} , which is independent of θ , such that

$$|w(0)| \leq \hat{c}|||w|||,$$

and thus

$$\begin{aligned} |||w|||^2 &\leq \left(3|Q_1| + \frac{1}{\beta - \alpha} + \frac{1}{|d_\theta|^2}\right) \int_{Q_1} |w'(y)|^2 dy \\ &\quad + 3 \left(\frac{1}{\beta - \alpha} + \frac{1}{|d_\theta|^2}\right) |\exp(2\pi i\theta) - 1|^2 \hat{c}^2 |||w|||^2. \end{aligned} \quad (3.5)$$

We now choose $\delta < 1/4$ so that $((\beta - \alpha)^{-1} + |d_\theta|^{-2}) |\exp(2\pi i\theta) - 1|^2 \hat{c}^2 < 1/2$, and hence $|d_\theta|^{-2}$ is bounded above by a constant independent of θ . The inequality (3.5) now immediately implies the required estimate.

Case III: $\theta \in [\delta, 1 - \delta]$. For given $x \in (\beta, 1], y \in [0, \alpha)$ we write

$$w(x) = \int_\beta^x w'(t) dt + w(\beta), \quad w(y) = - \int_y^\alpha w'(t) dt + w(\alpha),$$

which implies, in view of Proposition 3.4 (ii),

$$\begin{aligned} w(x) - w(y) &= w(\beta) - w(\alpha) + \left(\int_\beta^x + \int_y^\alpha\right) w'(t) dt \\ &= d_\theta \int_{Q_1} w(y) dy + \left(\int_\beta^x + \int_y^\alpha\right) w'(t) dt. \end{aligned}$$

In particular, substituting $x = 1, y = 0$ and using the fact that $w(1) = \exp(2\pi i\theta)w(0)$, we obtain

$$w(1) = \frac{d_\theta}{1 - \exp(-2\pi i\theta)} \int_{Q_1} w(y) dy + \frac{1}{1 - \exp(-2\pi i\theta)} \int_{Q_1} w'(y) dy,$$

whence

$$\begin{aligned} w(x) &= - \int_x^1 w'(t) dt + w(1) = - \int_x^1 w'(t) dt \\ &\quad + \frac{d_\theta}{1 - \exp(-2\pi i\theta)} \int_{Q_1} w(y) dy + \frac{1}{1 - \exp(-2\pi i\theta)} \int_{Q_1} w'(y) dy. \end{aligned}$$

Integrating the last identity over $(\beta, 1]$ yields

$$\begin{aligned} \int_\beta^1 w(y) dy &= - \int_\beta^1 \left(\int_x^1 w'(t) dt \right) dx \\ &\quad + \frac{(1 - \beta)d_\theta}{1 - \exp(-2\pi i\theta)} \int_{Q_1} w(y) dy + \frac{1 - \beta}{1 - \exp(-2\pi i\theta)} \int_{Q_1} w'(y) dy, \end{aligned} \quad (3.6)$$

Similarly, we write

$$\begin{aligned} w(y) &= \int_0^y w'(t) dt + \exp(-2\pi i\theta)w(1) \\ &= \int_0^y w'(t) dt + \frac{\exp(-2\pi i\theta)d_\theta}{1 - \exp(-2\pi i\theta)} \int_{Q_1} w(y) dy + \frac{\exp(-2\pi i\theta)}{1 - \exp(-2\pi i\theta)} \int_{Q_1} w'(y) dy, \end{aligned}$$

which upon integration over $(0, \alpha)$ yields

$$\begin{aligned} \int_0^\alpha w(y)dy &= \int_0^\alpha \left(\int_0^y w'(t)dt \right) dy \\ &+ \frac{\alpha \exp(-2\pi i \theta) d_\theta}{1 - \exp(-2\pi i \theta)} \int_{Q_1} w(y)dy + \frac{\alpha \exp(-2\pi i \theta)}{1 - \exp(-2\pi i \theta)} \int_{Q_1} w'(y)dy. \end{aligned} \quad (3.7)$$

Combining equations (3.6) and (3.7) we obtain

$$\begin{aligned} \left(1 - \frac{(1 - \beta + \alpha \exp(-2\pi i \theta)) d_\theta}{1 - \exp(-2\pi i \theta)} \right) \int_{Q_1} w(y)dy &= \int_0^\alpha \left(\int_0^y w'(t)dt \right) dy \\ &- \int_\beta^1 \left(\int_x^1 w'(t)dt \right) dx + \frac{(1 - \beta + \alpha \exp(-2\pi i \theta))}{1 - \exp(-2\pi i \theta)} \int_{Q_1} w'(y)dy. \end{aligned}$$

Squaring both sides and using the Cauchy-Schwarz inequality yields

$$\begin{aligned} \left| 1 - \frac{(1 - \beta + \alpha \exp(-2\pi i \theta)) d_\theta}{1 - \exp(-2\pi i \theta)} \right|^2 \left| \int_{Q_1} w(y)dy \right|^2 \\ \leq 2 \left(4 + \frac{|1 - \beta + \alpha \exp(-2\pi i \theta)|^2}{|1 - \exp(-2\pi i \theta)|^2} \right) \int_{Q_1} |w'(y)|^2 dy. \end{aligned}$$

A direct calculation shows that the coefficient in the left-hand side of the last inequality is separated from zero in the range of θ considered.

Finally, we argue that

$$\begin{aligned} |w(\beta) - w(\alpha)| &\leq 4(|w(\beta) - w(1)| + |w(0) - w(\alpha)| + |w(0)| + |w(1)|) \\ &= 4 \left(\left| \int_\beta^1 w'(y)dy \right| + \left| \int_0^\alpha w'(y)dy \right| + |w(0)| + |w(1)| \right), \end{aligned}$$

and

$$\begin{aligned} (1 - \beta)w(1) &= \int_\beta^1 \int_x^1 w'(t)dt dx + \int_\beta^1 w(x)dx, \\ \alpha w(0) &= - \int_0^\alpha \int_0^x w'(t)dt dx + \int_0^\alpha w(x)dx. \end{aligned}$$

The required inequality follows, since by (3.3) and Lemma 3.4 (i),

$$|||w||| = \left| \int_{Q_1} w(y)dy \right|^2 + \frac{|w(\beta) - w(\alpha)|^2}{\beta - \alpha} + \int_{Q_1} |w'(y)|^2 dy.$$

This completes the proof of Proposition 3.3. \square

We now resume the proof of Theorem 3.1.

Let N_ε be the smallest integer such that $\bar{\Omega} \subset \varepsilon[a/\varepsilon] + \cup_{n=0}^{N_\varepsilon-1} \varepsilon(n, n+1) =: \tilde{\Omega}^\varepsilon$. For each ε , we denote $D^\varepsilon := \varepsilon^{-1}\tilde{\Omega}^\varepsilon$ and use the rescaling operator T_ε defined by $(T_\varepsilon u)(y) = u(\varepsilon y)$, $y \in D^\varepsilon$, and an extension operator E_ε from \mathfrak{H} to $H^1(\tilde{\Omega}^\varepsilon)$ such that

$$\|E(u_\varepsilon)\|_{L^2(\tilde{\Omega}^\varepsilon)} \leq (1 + C_E \varepsilon) \|u_\varepsilon\|_{L^2(\Omega)}, \quad \|E(u_\varepsilon)'\|_{L^2(\tilde{\Omega}^\varepsilon)} \leq (1 + C_E \varepsilon) \|u_\varepsilon'\|_{L^2(\Omega)} \quad (3.8)$$

for all $\varepsilon > 0$, where the constant $C_E > 0$ is the same for all ε . (It is clear, for example, that the continuous extension of u_ε by constants satisfies this requirement.) We then define functions U_ε by the formula $U_\varepsilon := T_\varepsilon^{-1} P_{V(D^\varepsilon)} T_\varepsilon E(u_\varepsilon)$ for each value of ε .

Extending each of the functions U_ε to the whole of \mathbb{R} with period $\varepsilon N_\varepsilon$, we consider the “discrete Floquet-Bloch transform” of the functions U_ε as follows

$$U_\varepsilon^j(x) = \frac{1}{N_\varepsilon} \sum_{k=0}^{N_\varepsilon-1} U_\varepsilon(x + \varepsilon k) \exp(2\pi i j k / N_\varepsilon), \quad x \in \tilde{\Omega}_\varepsilon, \quad j = 0, 1, \dots, N_\varepsilon - 1,$$

so that $T_\varepsilon U_\varepsilon^j \in V(j/N_\varepsilon)$ and $U_\varepsilon = \sum_{j=0}^{N_\varepsilon-1} U_\varepsilon^j$. Using the fact that the eigenfunctions $v^k(j/N_\varepsilon)$, $k \in \mathbb{N}$, form a complete system in the $L^2(Q)$ -closure of the set $V(j/N_\varepsilon)$, we represent U_ε^j in terms of them so that

$$U_\varepsilon(x) = \sum_{j=0}^{N_\varepsilon-1} \sum_{k=1}^{\infty} \hat{U}_\varepsilon^k\left(\frac{j}{N_\varepsilon}\right) v^k\left(\frac{j}{N_\varepsilon}, \frac{x}{\varepsilon}\right), \quad (3.9)$$

$$U'_\varepsilon(x) = \varepsilon^{-1} \sum_{j=0}^{N_\varepsilon-1} \sum_{k=1}^{\infty} \hat{U}_\varepsilon^k\left(\frac{j}{N_\varepsilon}\right) (v^k)' \left(\frac{j}{N_\varepsilon}, \frac{x}{\varepsilon}\right), \quad (3.10)$$

where $\hat{U}_\varepsilon^k(j/N_\varepsilon) \in \mathbb{C}$. Applying the Parseval identity to (3.9), followed by Proposition 3.2 and the first of the *a priori* bounds (3.2) yields

$$\varepsilon N_\varepsilon \sum_{j=0}^{N_\varepsilon-1} \sum_{k=1}^{\infty} \left| \hat{U}_\varepsilon^k\left(\frac{j}{N_\varepsilon}\right) \right|^2 = \|U_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)}^2 \geq (1 - C_\perp C_B \varepsilon) \|u_\varepsilon\|_{L^2(\Omega)}^2.$$

The last estimate, in combination with the first inequality in (3.8) and the fact that $\|u_\varepsilon\|_{L^2(\Omega)} = 1$, implies

$$1 - C_\perp C_B \varepsilon \leq \varepsilon N_\varepsilon \sum_{j=0}^{N_\varepsilon-1} \sum_{k=1}^{\infty} \left| \hat{U}_\varepsilon^k\left(\frac{j}{N_\varepsilon}\right) \right|^2 \leq 1 + C_E \varepsilon \quad (3.11)$$

Denoting by $\delta(\cdot - \theta)$ the Dirac mass at θ , we infer from (3.11) the existence of $C > 0$ such that

$$\left| \sum_{k=1}^{\infty} \int_0^1 d\mu_\varepsilon^k - 1 \right| \leq C\varepsilon, \quad d\mu_\varepsilon^k(\theta) := \varepsilon N_\varepsilon \sum_{j=0}^{N_\varepsilon-1} \left| \hat{U}_\varepsilon^k\left(\frac{j}{N_\varepsilon}\right) \right|^2 \delta\left(\theta - \frac{j}{N_\varepsilon}\right) d\theta. \quad (3.12)$$

Clearly, for each k the sequence $\{\mu_\varepsilon^k\}_\varepsilon$ is bounded in the space of Radon measures on $[0, 1]$. Therefore, up to a subsequence, μ_ε^k weakly converge as $\varepsilon \rightarrow 0$ to some measure μ^k :

$$\int_0^1 u \, d\mu_\varepsilon^k \rightarrow \int_0^1 u \, d\mu^k, \quad \forall u \in C[0, 1]. \quad (3.13)$$

The above result follows from recalling that the space of finite Radon measures on $[0, 1]$ coincides with the dual space $C[0, 1]^*$ and hence bounded sets of Radon measures and relatively compact with respect to weak star convergence in this space.

Furthermore, taking into account (3.10), the second of the estimates (3.8) and the second of the *a priori* bounds (3.2), we write

$$C_B(1 + C_E\varepsilon) \geq \varepsilon^2 \|U'_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)}^2 = \varepsilon N_\varepsilon \sum_{j=0}^{N_\varepsilon-1} \sum_{k=1}^{\infty} \left| \hat{U}_\varepsilon^k\left(\frac{j}{N_\varepsilon}\right) \right|^2 \left\| (v^k)' \left(\frac{j}{N_\varepsilon}, \cdot \right) \right\|_{L^2(Q)}^2 \quad (3.14)$$

$$\geq \|p^{-1}\|_{L^\infty(Q)} \varepsilon N_\varepsilon \sum_{j=0}^{N_\varepsilon-1} \sum_{k=1}^{\infty} \left| \hat{U}_\varepsilon^k\left(\frac{j}{N_\varepsilon}\right) \right|^2 \lambda_k\left(\frac{j}{N_\varepsilon}\right) \quad (3.15)$$

$$= \|p^{-1}\|_{L^\infty(Q)} \sum_{k=1}^{\infty} \int_0^1 \lambda_k(\theta) d\mu_\varepsilon^k(\theta). \quad (3.16)$$

In order to obtain (3.15), we use the fact that $v^k(\theta)$, $k \in \mathbb{N}$, are eigenfunctions of the operator $A(\theta)$ with eigenvalues $\lambda_k(\theta)$, and (3.16) is a version of the same expression written in terms of the measures defined in (3.12). For any integer $K \geq 2$ the inequality (3.14)–(3.16) immediately implies

$$\sum_{k=K}^{\infty} \int_0^1 d\mu_\varepsilon^k(\theta) \leq \|p^{-1}\|_{L^\infty(Q)}^{-1} C_B(1 + C_E\varepsilon) \min_{\theta \in [0,1]} \lambda_K(\theta)^{-1} \xrightarrow{K \rightarrow \infty} 0.$$

In view of (3.12) and (3.13) we thus argue that $\sum_{k=1}^{\infty} \mu^k([0, 1]) = 1$, hence μ^k is a non-zero measure for some k .

We next show that $\lambda = \lambda_{k_0}(\theta)$ for some k_0, θ . To this end we set $\varphi = U_\varepsilon$ in the weak formulation (3.1):

$$\int_{\Omega_1^\varepsilon} p\left(\frac{x}{\varepsilon}\right) u'_\varepsilon(x) \overline{U'_\varepsilon(x)} dx + \varepsilon^2 \int_{\Omega_0^\varepsilon} p\left(\frac{x}{\varepsilon}\right) u'_\varepsilon(x) \overline{U'_\varepsilon(x)} dx = \lambda_\varepsilon \int_{\Omega} u_\varepsilon(x) \overline{U_\varepsilon(x)} dx. \quad (3.17)$$

Notice that, by the definition of the space $V(D^\varepsilon)$, the first term in the equation (3.17) vanishes. We also have, in view of Proposition 3.2 and the second of the estimates (3.2), as $\varepsilon \rightarrow 0$:

$$\begin{aligned} \varepsilon^2 \int_{\Omega_0^\varepsilon} p\left(\frac{x}{\varepsilon}\right) u'_\varepsilon(x) \overline{U'_\varepsilon(x)} dx &= \varepsilon^2 \int_{\Omega_0^\varepsilon} p\left(\frac{x}{\varepsilon}\right) U'_\varepsilon(x) \overline{U'_\varepsilon(x)} dx + o(1) \\ &= \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \sum_{j=0}^{N_\varepsilon-1} \int_{\varepsilon Q_0} p\left(\frac{x}{\varepsilon}\right) \hat{U}_\varepsilon^k\left(\frac{j}{N_\varepsilon}\right) (v^k)' \left(\frac{j}{N_\varepsilon}, \frac{x}{\varepsilon} \right) \overline{\hat{U}_\varepsilon^l\left(\frac{j}{N_\varepsilon}\right) (v^l)' \left(\frac{j}{N_\varepsilon}, \frac{x}{\varepsilon} \right)} dx + o(1) \\ &= \varepsilon N_\varepsilon \sum_{k=1}^{\infty} \sum_{j=0}^{N_\varepsilon-1} \lambda_k\left(\frac{j}{N_\varepsilon}\right) \left| \hat{U}_\varepsilon^k\left(\frac{j}{N_\varepsilon}\right) \right|^2 + o(1) = \sum_{k=1}^{\infty} \int_0^1 \lambda_k(\theta) d\mu_\varepsilon^k + o(1). \end{aligned} \quad (3.18)$$

In the last three equalities of (3.18) we use (3.9), (3.12) and the fact that $v^k(\theta)$ are orthogonal eigenfunctions of $A(\theta)$ with eigenvalues $\lambda_k(\theta)$. Similarly we have

$$\int_0^1 u_\varepsilon(x) \overline{U_\varepsilon(x)} dx = \sum_{k=1}^{\infty} d\mu_\varepsilon^k + o(1), \quad (3.19)$$

as $\varepsilon \rightarrow 0$. Combining (3.17), (3.18), (3.19) yields

$$\sum_{k=1}^{\infty} \int_0^1 \lambda_k(\theta) d\mu_\varepsilon^k(\theta) = \lambda_\varepsilon \sum_{k=1}^{\infty} \int_0^1 d\mu_\varepsilon^k + o(1). \quad (3.20)$$

Finally, passing to the limit $\varepsilon \rightarrow 0$ in (3.20) and using the fact λ_k are continuous functions of θ (see Appendix C) yields

$$\sum_{k=1}^{\infty} \int_0^1 \lambda_k(\theta) d\mu^k(\theta) = \lambda \sum_{k=1}^{\infty} \int_0^1 d\mu^k.$$

We already know that μ^{k_0} is a non-zero measure for some k_0 , hence $\lambda = \lambda_{k_0}(\theta)$ for some θ .

4. A modified problem with a compact perturbation, and the associated defect modes.

4.1. Analytical setup. In this section we discuss a modified version of the setup of Section 1.2, as follows. Consider the operator \tilde{A}^ε defined by the bilinear form

$$(\tilde{A}^\varepsilon u, v) = \int_a^b \left(p_d \chi_d(x) + p(x/\varepsilon) (\varepsilon^2 \chi_0(x/\varepsilon) + \chi_1(x/\varepsilon)) (1 - \chi_d(x)) \right) u'(x) v'(x) dx, \quad (4.1)$$

$u, v \in \mathfrak{H}$. Here the space \mathfrak{H} is as before, and χ_d is the indicator function of a “defect” interval I_d whose closure is assumed to be contained in (a, b) , and p_d is the corresponding “defect” coefficient (or “defect strength”). Analogously to the above, we assume that a, b , and the end-points of I_d belong to the set εF_1 , otherwise we modify (4.1) on those connected components of εF_0 that contain a, b and the end-points of I_d by changing the related coefficient from ε^2 to the unity. We denote by \tilde{S}^ε the spectrum of the operator \tilde{A}^ε .

A formal two-scale asymptotic procedure carried out on the equation (cf. (1.1)) $\tilde{A}^\varepsilon u = \lambda u$ suggests that:

- 1) The set $\lim_{\varepsilon \rightarrow 0} \tilde{S}^\varepsilon$ is independent of the choice of the space \mathfrak{H} and is given by the union of solutions to the equation (2.1) for all $\theta \in [0, 1)$ and a sequence of “defect eigenvalues” $\{p_d \pi^2 j^2 / |I_d|^2\}_{j=1}^{\infty}$;
- 2) The “defect eigenfunctions” u_j corresponding to the above eigenvalues, $j \in \mathbb{N}$, decay exponentially away from the boundary of the defect:

$$|u_j(x)| \leq C \exp(-\varepsilon^{-1} \text{dist}\{x, I_d\}), \quad x \in (a, b) \setminus I_d, \quad C > 0.$$

We next present numerical evidence that supports these claims.

4.2. Numerical results for the modified problem. We consider a defect of length $|I_d| = 1/2$ and strength $p_d = 2$ in the middle of the interval $(a, b) = (0, 1)$, i.e. $I_d = (1/4, 3/4)$. For each value of ε such that $N := \varepsilon^{-1}$ is a positive integer, we describe the intervals $(0, 1/4)$ and $(3/4, 1)$ on either sides of the defect according to

the equation (4.1): each of them consists of N cells of the same length $1/(4N)$, and in one half of each cell the coefficient in the form (the “modulus”) takes the value $1/(4N)^2$ while in the other half it is equal to unity. Importantly, we assume that the endpoints both of the interval (a, b) and of the defect are belong to the material component where the modulus is equal to unity.

The results of solving the above problems with finite elements are given in Tables 4.1 and 4.2. The values for the trapped mode are in good agreement with the values obtained by the asymptotic method: $\lambda_\star^{(2)} = 78.9568$, $\lambda_\star^{(3)} = 315.8273$, $\lambda_\star^{(5)} = 710.6115$, $\lambda_\star^{(6)} = 1263.3094$, where the superscript is the number of the stop band containing the related eigenvalue.

In addition, the profiles obtained for such trapped modes (see Figure 4.1 for the case of periodic boundary conditions) suggest that the number of half-oscillations in a trapped mode is equal to the number of the mode in the sequence, which resembles the behaviour of the usual Neumann eigenfunctions on the defect. We also note that the decay of the trapped modes appears to be exponential, as can be seen in Figure 4.2: the larger the contrast (and hence the number of subdivisions of the string) the more localised the mode, irrespective of the boundary conditions at the endpoints of the string.

Dirichlet boundary conditions					
$\lambda_{\min}^{(k,128)}$	$\lambda_{\max}^{(k,128)}$	$\lambda_\star^{(k,128)}$	$\lambda_\star^{(k,256)}$	$\lambda_\star^{(k,512)}$	$\lambda_\star^{(k,1024)}$
11.7939	39.4603	-	-	-	-
65.7875	157.8859	75.7674	77.2502	78.0741	78.7304
187.6799	355.2599	293.9534	304.1141	309.7163	314.2461
386.1413	622.2747	-	-	-	-
662.9213	986.7685	682.6577	694.4984	702.0486	708.4576
1018.4394	1421.0468	1225.1298	1232.2190	1243.1182	1258.2799

TABLE 4.1

Stop bands and trapped modes for the modified problem with a defect, subject to the Dirichlet boundary conditions: $\lambda_{\min}^{(k,128)}$ and $\lambda_{\max}^{(k,128)}$ are the lower and upper bounds of the k^{th} stop band for $N = 128$, and $\lambda_\star^{(k,128)}$, $\lambda_\star^{(k,256)}$, $\lambda_\star^{(k,512)}$, $\lambda_\star^{(k,1024)}$ are the trapped-mode eigenvalues in the k^{th} stop band, evaluated for $N = 128$, $N = 256$, $N = 512$ and $N = 1024$ respectively.

Neumann boundary conditions					
$\lambda_{\min}^{(k,128)}$	$\lambda_{\max}^{(k,128)}$	$\lambda_\star^{(k,128)}$	$\lambda_\star^{(k,256)}$	$\lambda_\star^{(k,512)}$	$\lambda_\star^{(k,1024)}$
11.7515	39.4980	-	-	-	-
65.8359	157.8901	75.7676	77.2509	78.0741	78.7314
187.7334	355.2765	293.9539	304.1145	309.7164	314.3057
386.2091	622.2747	-	-	-	-
662.9779	986.7698	682.6578	694.4985	702.0486	708.6496
1018.5163	1421.0556	1225.1298	1232.2190	1243.1193	1258.2626

TABLE 4.2

Stop bands and trapped modes for the modified problem with a defect, subject to the Neumann boundary conditions: $\lambda_{\min}^{(k,128)}$ and $\lambda_{\max}^{(k,128)}$ are the lower and upper bounds of the k^{th} stop band for $N = 128$, and $\lambda_\star^{(k,128)}$, $\lambda_\star^{(k,256)}$, $\lambda_\star^{(k,512)}$, $\lambda_\star^{(k,1024)}$ are the trapped-mode eigenvalues in the k^{th} stop band, evaluated for $N = 128$, $N = 256$, $N = 512$ and $N = 1024$ respectively.

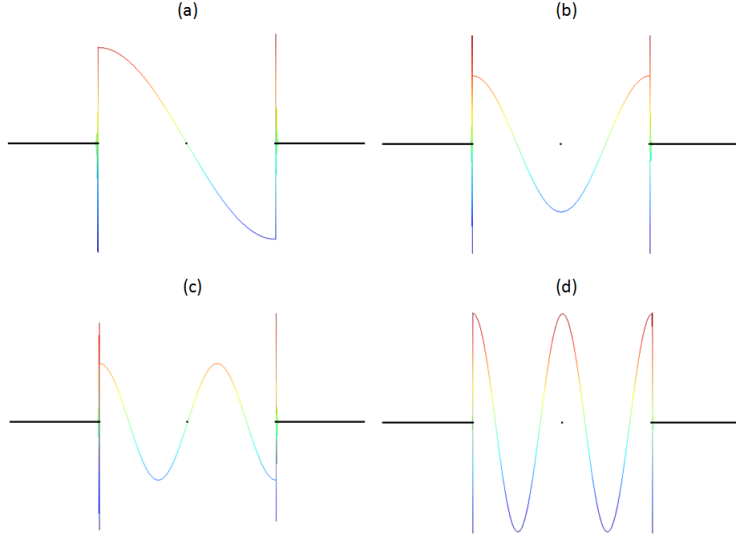


FIG. 4.1. Trapped eigenmodes for the modified problem with a defect and periodic boundary conditions ($N=1024$), corresponding to eigenfrequencies: (a) $\lambda_{\star}^{(2,1024)} = 78.07$, (b) $\lambda_{\star}^{(3,1024)} = 314.24$, (c) $\lambda_{\star}^{(5,1024)} = 708.46$, (d) $\lambda_{\star}^{(6,1024)} = 1258.28$.

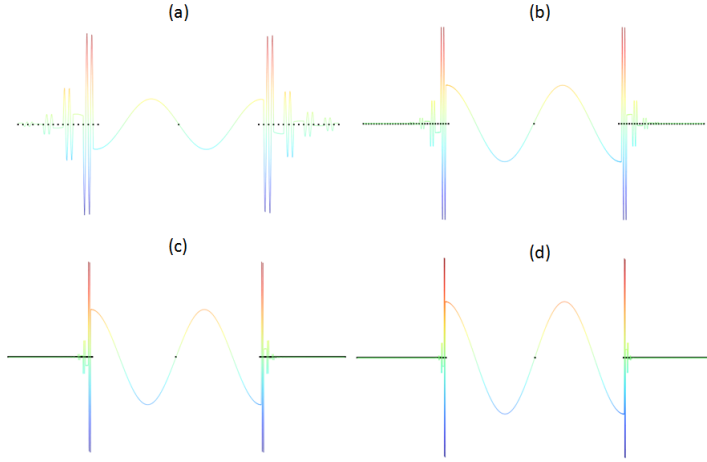


FIG. 4.2. Decay of the third trapped eigenmode (located in the fifth stop band) for the modified problem with a defect and periodic boundary conditions, as a function of contrast: (a) $\lambda_{\star}^{(5,32)} = 668.86$, (b) $\lambda_{\star}^{(5,64)} = 682.89$, (c) $\lambda_{\star}^{(5,128)} = 694.50$, (d) $\lambda_{\star}^{(5,256)} = 702.68$.

4.3. Photonic band gaps and trapped modes in high-contrast multi-layered dielectric structures. The string problem emerges, among other contexts, in the study of wave propagation in one-dimensional photonic crystals *i.e.* multi-layered dielectric structures invariant along two directions. In what follows we set these directions to be x_1 and x_3 in the usual Euclidean representation $x = (x_1, x_2, x_3)$.

We consider those solutions $(\mathcal{E}, \mathcal{H})$ to the classical system of Maxwell equations ([9]) that have the form

$$\mathcal{E}(x_1, x_2, x_3, t) = E(x_2) \exp(i(\kappa x_3 - \omega t)), \quad \mathcal{H}(x_1, x_2, x_3, t) = H(x_2) \exp(i(\kappa x_3 - \omega t)),$$

where t is time, ω is the angular frequency, and $\kappa \geq 0$ is a “propagation constant”. We write the Maxwell equations for the field variable (E, H) :

$$\begin{cases} E'_3 - i\kappa E_2 &= i\omega\mu H_1, \\ i\kappa E_1 &= i\omega\mu H_2, \\ -E'_1 &= i\omega\mu H_3, \end{cases} \quad \begin{cases} -H'_3 + i\kappa H_2 &= i\omega\varepsilon E_1, \\ -i\kappa H_1 &= i\omega\varepsilon E_2, \\ H'_1 &= i\omega\varepsilon E_3, \end{cases}$$

Here μ is the magnetic permeability, ε is the electric permittivity at each point of the dielectric.

We rearrange the above six equations in two groups of equations for (E_1, H_2, H_3) (transverse magnetic polarisation), and (H_1, E_2, E_3) (transverse electric polarisation). We choose E_1 and H_1 as the unknown functions within the respective groups and notice that the remaining unknowns are expressed in terms of these two scalar functions only. The equations satisfied by E_1, H_1 are

$$(E'_1)' + (\omega^2\mu\varepsilon - \kappa^2)E_1 = 0, \quad (4.2)$$

$$(\varepsilon^{-1}H'_1)' + (\omega^2\mu - \varepsilon^{-1}\kappa^2)H_1 = 0. \quad (4.3)$$

Note that (4.3) coincides with (1.1) when $\kappa = 0$, by setting

$$\omega^2 = \lambda, \quad \mu = 1, \quad \varepsilon^{-1}(x_2) = p(x_2/\eta)(\eta^2\chi_0(x_2/\eta) + \chi_1(x_2/\eta)), \quad x_2 \in (a, b), \quad \eta > 0,$$

where we use η rather than ε to denote the structure period, in order to avoid confusion with the standard notation for electric permittivity. Our analysis in Sections 2 and 3 carries over to the case $\kappa > 0$, where we get a κ -dependent version of the dispersion relation (2.1), as follows:

$$\frac{1}{2}(\alpha - \beta + 1) \left(\sqrt{\lambda} - \frac{\kappa^2}{\sqrt{\lambda}} \right) \sin(\sqrt{\lambda}(\alpha - \beta)) + \cos(\sqrt{\lambda}(\alpha - \beta)) = \cos(2\pi\theta).$$

Assuming infinitely conducting walls on either side of the dielectric (see *e.g.* [20] for further details), we supply (4.2) and (4.3) with homogeneous Dirichlet and Neumann boundary conditions respectively.

In the numerical solution of above problem we employ finite elements with perfectly matched layers, *i.e.* anisotropic absorptive reflectionless layers (see *e.g.* [20]), on the top and bottom of the computational domain. Our results are shown in Figure 4.3 for $\kappa = .1$, $N = 16$, and for the transverse electric mode with frequency $\lambda_\star = 78.34$ inside the third stop band. The latter corresponds to the first trapped mode shown in Figure 4.1(a), in view of the fact that for $\kappa = .1$ there is an additional zero-frequency stop band. The magnetic component of this mode (Figures 4.3(a) and (d)) clearly shares the same features as the string mode in Figure 4.1(a).

Appendix A. In this appendix we argue that for any fixed $N \in \mathbb{N}$ and $\lambda < 0$, say $\lambda = -1$, the solutions to the problems (1.1) converge in an appropriate two-scale sense (see *e.g.* [1]) to the solution of some limit problem parametrised by N . Throughout our argument, we assume that $(a, b) = \mathbb{R}$, although the results are valid for an arbitrary interval $(a, b) \subset \mathbb{R}$, irrespective of the boundary conditions.

We first formulate the related statement for $N = 1$.

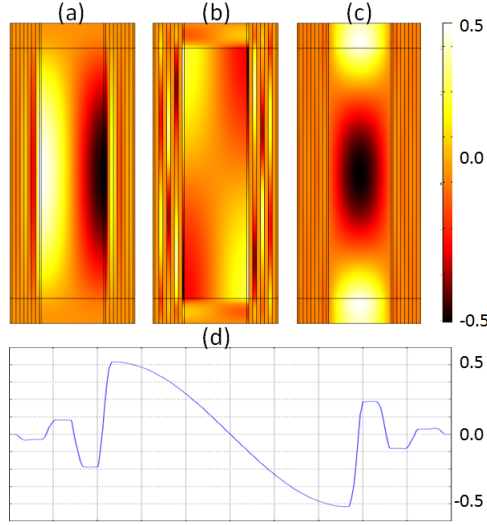


FIG. 4.3. Obliquely propagating transverse electric wave in a high-contrast dielectric multilayered planar waveguide with infinitely conducting walls: (a) 2D plot of H_1 ; (b) 2D plot of E_2 ; (c) 2D plot of E_3 ; (d) Profile of H_1 along the horizontal centreline. Here $N = 16$, $\kappa = .1$ and the normalized frequency $\lambda_\star = 78.34$.

4.4. Periodic homogenisation. LEMMA 4.1. Set $\lambda = -1$ and let $u_\varepsilon \in H^1(\mathbb{R})$ be the solution to the problem (1.1). Then there exists $u(x, y) \in L^2(\Omega; V_1)$ such that

$$u_\varepsilon \xrightarrow{2} u(x, y), \quad \varepsilon u'_\varepsilon \xrightarrow{2} u_{,y}(x, y), \quad \chi_1\left(\frac{x}{\varepsilon}\right) p\left(\frac{x}{\varepsilon}\right) u'_\varepsilon(x) \xrightarrow{2} 0.$$

Here the space V_1 is defined by

$$V_1 := \{v \in H^1_\#(Q) : p(y)v'(y) = 0 \text{ for } y \in Q_1\}. \quad (4.4)$$

The function $u \in L^2(\mathbb{R}; V_1)$ is the unique solution to the problem

$$\begin{aligned} & \int_{\mathbb{R}} \int_Q p(y) u_{,y}(x, y) \overline{\varphi'(y) \psi(x)} \, dy dx + \int_{\mathbb{R}} \int_Q u(x, y) \overline{\varphi(y) \psi(x)} \, dy dx \\ &= \int_{\Omega} \int_Q f(x) \overline{\varphi(y) \psi(x)} \, dy dx, \quad \forall \varphi \in C_0^\infty(\mathbb{R}), \psi \in V_1. \end{aligned} \quad (4.5)$$

Note that the x dependence in the formulation (4.5) is trivial, indeed $u(x, y) = f(x)v(y)$ where $v(y) \in V_1$ is the unique solution to the problem

$$\int_{Q_0} p(y) v'(y) \overline{\varphi'(y)} dy + \int_Q v(y) \overline{\varphi(y)} dy = \int_Q \overline{\varphi(y)} dy, \quad \forall \varphi \in V_1. \quad (4.6)$$

This observation implies, in particular, that the spectra corresponding to (4.5) and (4.6) coincide. Therefore, denoting by A_1 the operator defined by the form

$$\mathbf{b}_1(u, v) = \int_{Q_0} p(y) u'(y) \overline{v'(y)} dy, \quad u, v \in V_1,$$

we obtain $\lim_{\varepsilon \rightarrow 0} S^\varepsilon \supset \sigma(A_1)$.

4.5. “ NQ -periodic” homogenisation. In the argument above we found the two-scale limit operator A_1 by choosing the periodic reference cell to be Q and passing to the two-scale limit in (1.1) as $\varepsilon \rightarrow 0$. Replacing Q with NQ , $N \in \mathbb{N}$, we obtain an analogue of Lemma 4.1, as follows.

LEMMA 4.2. *Set $\lambda = -1$ and let u_ε be the solution to (1.1). Then $u_\varepsilon \xrightarrow{2} u_N$, up to some sequence we do not relabel, where $u_N = f(x)v_N(y)$ and v_N is the unique solution to*

$$\int_{NQ} \chi_0(y)p(y)(v_N)'(y)\overline{\varphi'(y)}dy + \int_{NQ} v_N(y)\overline{\varphi(y)}dy = \int_{NQ} \overline{\varphi(y)}dy, \quad \forall \varphi \in V_N.$$

Here we denote

$$V_N := \{v \in H_{\#}^1(NQ) : p(y)v'(y) = 0 \text{ for } y \in Q_1\}. \quad (4.7)$$

Furthermore, $\lim_{\varepsilon \rightarrow 0} S^\varepsilon \supset \sigma(A_N)$, where A_N is the operator defined using the bilinear form

$$\mathbf{b}_N(u, v) := \int_{NQ} \chi_0(y)p(y)u'(y)\overline{v'(y)}dy, \quad u, v \in H_{\#}^1(NQ). \quad (4.8)$$

Applying Lemma 4.2 for all $N \in \mathbb{N}$ yields

$$\lim_{\varepsilon \rightarrow 0} \sigma(A_\varepsilon) \supset \bigcup_{N \in \mathbb{N}} \sigma(A_N).$$

4.6. Relation to the Bloch spectrum. Notice that if $\theta = j/N$ for some integers N, j such that $0 \leq j \leq N-1$, then all eigenfunctions $v^k(\theta)$, $k = 1, 2, \dots$, are N -periodic, in particular $v^k(\theta) \in V_N$. In fact, $v^k(\theta)$, $k = 1, 2, \dots$ are eigenfunctions of A_N . Indeed, for any fixed $\varphi \in V_N$ define the function $\Phi(y) := \sum_{k=0}^{N-1} \varphi(y+k) \exp(-2\pi i \theta k)$ and notice that since $\Phi(y) \in V(\theta)$, one has $(A(\theta)v^k(\theta), \Phi) = \lambda_k(\theta)(v^k(\theta), \Phi)$. Therefore, writing for brevity $v^k(\theta) = v$, we obtain

$$\begin{aligned} \int_{NQ} \chi_0(y)p(y)v'(y)\overline{\varphi'(y)}dy &= \sum_{k=0}^{N-1} \int_Q \chi_0(y+k)p(y+k)v'(y+k)\overline{\varphi'(y+k)}dy \\ &= \int_Q \chi_0(y)p(y)v(y)\overline{\Phi(y)}dy = \lambda_k(\theta) \int_Q v(y)\overline{\Phi(y)}dy \\ &= \sum_{k=0}^{N-1} \lambda_k(\theta) \int_Q v(y) \exp(2\pi i \theta k) \overline{\varphi(y+k)}dy = \lambda_k(\theta) \int_{NQ} v(y)\overline{\varphi(y)}dy. \end{aligned}$$

The above observations show that

$$\lim_{\varepsilon \rightarrow 0} S^\varepsilon \supset \bigcup_{N \in \mathbb{N}} \sigma(A_N) \supset \bigcup_{\substack{N \in \mathbb{N} \\ 0 \leq j \leq N-1}} \sigma(A(j/N)).$$

Using the facts that the set of rational numbers j/N is dense in $[0, 1)$ and that the eigenvalues $\lambda = \lambda(\theta)$ are continuous with respect to θ (see Appendix C below) yields

$$\lim_{\varepsilon \rightarrow 0} S^\varepsilon \supset \bigcup_{\theta \in [0, 1)} \sigma(A(\theta)).$$

Appendix B.

4.7. One classical inequality. Here, for reader's convenience, we give the proof of a version of the classical Poincaré inequality (see *e.g.* [6]), which we use in the present paper.

LEMMA 4.3. *There exists a positive constant C_P , which depends on α and β only, such that*

$$\int_Q |u(y)|^2 dy \leq C_P \left(\left| \int_{Q_1} u(y) dy \right|^2 + \int_Q |u'(y)|^2 dy \right), \quad \forall u \in H^1(Q). \quad (4.9)$$

Proof. For fixed $x \in Q$, $y \in Q_1$, we have $|x - y| \leq 1$ and

$$u(x) - u(y) = \int_y^x u'(t) dt,$$

which implies

$$|u(x)|^2 + |u(y)|^2 - 2\Re(u(x)\overline{u(y)}) = \left| \int_y^x u'(t) dt \right|^2 \leq |x - y| \int_y^x |u'(t)|^2 dt \leq \int_Q |u'(y)|^2 dy.$$

Integrating first with respect to x , then with respect to y , yields

$$\begin{aligned} |Q_1| \int_Q |u(x)|^2 dx + \int_{Q_1} |u(y)|^2 dy \\ \leq 2\Re \left[\left(\int_Q u(x) dx \right) \overline{\left(\int_{Q_1} u(y) dy \right)} \right] + |Q_1| \int_Q |u'(y)|^2 dy. \end{aligned}$$

Here we have used the fact that $|Q| = 1$. Now using the inequalities (4.7), $ab \leq \varepsilon a^2 + b^2/\varepsilon$ for any real a, b , $\varepsilon > 0$, and

$$\left| \int_{Q_1} u(y) dy \right|^2 \leq |Q_1| \int_{Q_1} |u(y)|^2 dy,$$

we obtain

$$\begin{aligned} |Q_1| \int_Q |u(x)|^2 dx + \frac{1}{|Q_1|} \left| \int_{Q_1} u(y) dy \right|^2 &\leq |Q_1| \int_Q |u(x)|^2 dx + \int_{Q_1} |u(y)|^2 dy \\ &\leq \varepsilon \left(\int_Q |u(y)|^2 dy \right) + \frac{1}{\varepsilon} \left| \int_{Q_1} u(y) dy \right|^2 + |Q_1| \int_Q |u'(y)|^2 dy. \end{aligned}$$

Setting $\varepsilon = |Q_1|/2$ gives

$$\frac{|Q_1|}{2} \int_Q |u(x)|^2 dx \leq \frac{1}{|Q_1|} \left| \int_{Q_1} u(y) dy \right|^2 + |Q_1| \int_Q |u'(y)|^2 dy.$$

This is (4.9) for $C_P = 2/|Q_1|^2$, where $|Q_1| = 1 - \beta + \alpha$. \square

4.8. The equivalence of Proposition 3.2 and Proposition 3.3. Suppose that Proposition 3.3 holds and consider a bounded interval $D \subset \mathbb{R}$ and a function $u \in V(D)^\perp$. For the proof of Theorem 3.1 it is sufficient to consider the case when D is an interval of integer length. However, for completeness we carry out the argument for an arbitrary D .

Clearly, there is a positive integer N and an interval $I := (l, l + N) \supset \overline{D}$, $l \in \mathbb{R}$ such that $\{l, l + 1\} \subset D_1$. We extend the function u to a function that is periodic on the interval I and is such that $u \in V(I)^\perp$, keeping the same notation u for such an extension.

Notice that for each $j = 0, 1, \dots, N - 1$ the function

$$u^j(x) = N^{-1} \sum_{k=1}^N u(x - l + k) \exp(-2\pi i j k / N)$$

belongs to the space $V(\theta_j)^\perp$, $\theta_j := j/N$. Indeed, for any $v \in V(\theta_j) \subset H_{\theta_j}^1(Q)$ one has

$$v(x) = v_\#(x) \exp(i\theta_j x), \quad (4.10)$$

where $v_\# \in H_\#(Q)$. We extend $v_\#$ by periodicity to the whole of \mathbb{R} and also extend v to the whole of \mathbb{R} so that the formula (4.10) holds for any $x \in \mathbb{R}$. Then, for the extended function v , one clearly has $v(\cdot + l) \in V(I)$, and $v(x - k) = v(x) \exp(-2\pi i j k / N)$ for any $x \in \mathbb{R}$. Therefore, for any $j = 0, 1, \dots, N - 1$, one has

$$\begin{aligned} \int_Q u^j(x) \overline{v(x)} dx &= N^{-1} \sum_{k=0}^{N-1} \int_0^1 u(x - l + k) \exp(-2\pi i j k / N) \overline{v(x)} dx \\ &= N^{-1} \sum_{k=0}^{N-1} \int_k^{k+1} u(x) \exp(-2\pi i j k / N) \overline{v(x - k)} dx \\ &= N^{-1} \int_0^N u(x - l) \overline{v(x)} dx = N^{-1} \int_I u(x) \overline{v(x + l)} dx = 0. \end{aligned}$$

Now, using the Parseval identity and Lemma 4.3, we obtain

$$\begin{aligned} \|u\|_{H^1(D)}^2 &\leq \|u\|_{H^1(I)}^2 = N \sum_{j=1}^N \|u^j\|_{H^1(Q)}^2 \leq N(C_P + 1) \sum_{j=1}^N \|u^j\|^2 \\ &\leq (C_P + 1) \tilde{C}^2 N \sum_{j=1}^N \|(u^j)'\|_{L^2(Q_1)}^2 = (C_P + 1) \tilde{C}^2 \|u'\|_{L^2(I_1)}^2, \end{aligned}$$

where $I_1 := I \cap F_1$. By ensuring, when performing the above extension from D to I , that the inequality $\|u'\|_{L^2(I_1)}^2 \leq C \|u'\|_{L^2(D_1)}^2$ holds with a positive constant C that does not depend on the interval D , we complete the proof of the implication.

Conversely, assume the validity of Proposition 3.2 and suppose that $\theta \in [0, 1)$ is rational, $N\theta \in \mathbb{N}$. For any $w \in V(\theta)^\perp$, we notice that

$$w(x) = u(x) \exp(i\theta x) \quad (4.11)$$

for some $u \in H_\theta^1(Q)$, and extend first the function u by periodicity to the interval $D = NQ$ and then the function w according to the formula (4.11) to the same interval D . Then using an argument similar to the above it is shown that $w \in V(D)^\perp$, and hence

$$\|w\|_{H^1(D)} \leq C \|w'\|_{L^2(D_1)}$$

for some positive constant C . This automatically implies (3.4) since the norm of w is the same for any interval of length one. By continuity in θ the statement is extended to establish the existence of $C > 0$ that serves all $\theta \in [0, 1)$.

5. Appendix C. The continuity of the family $V(\theta)$ implies that for a given k dimensional subspace of $V(\theta)$ we can find, for θ' close to θ , a k dimensional subspace of $V(\theta')$ close to $V(\theta)$, in the sense of [11]. More precisely,

PROPOSITION 5.1. *Let $\theta_1 \in [0, 1)$ and $F_1 \subset V(\theta_1)$, $\dim F_1 = k$. For all $\varepsilon > 0$ there exists $\delta > 0$ such that for all θ_2 , $|\theta_2 - \theta_1| \leq \delta$, there exists $F_2 \subset V(\theta_2)$, $\dim F_2 = k$ such that $\max\{\text{dist}(F_1, F_2), \text{dist}(F_2, F_1)\} < \varepsilon$, where the distance between two linear subspaces N, M , of $H^1(Q)$, is defined by the formula*

$$\text{dist}(N, M) := \sup_{\substack{u \in N \\ \|u\|_{H^1} = 1}} \inf_{v \in M} \|u - v\|.$$

Henceforth within this appendix, we write H^1 instead of $H^1(Q)$ for brevity.

Proof. Let $F_1 \subset V(\theta_1)$, $\dim F_1 = k$, and let f_n^1 , $n = 1, \dots, k$, be a basis of F_1 . Lemma 2.2 implies that for any θ_2 that is sufficiently close to θ_1 , there exist $v_n \in V(\theta_2)$ that are close to f_n^1 in the H^1 -norm. We construct an orthonormal sequence as follows:

$$f_1^2 = v_1, \quad f_2^2 = a_{21}f_1^2 + v_2, \quad f_n^2 = \sum_{m=1}^{n-1} a_{nm}f_m^2 + v_n, \quad n = 3, \dots, k,$$

where $a_{nm} = -\|f_m^2\|_{H^1}^{-1} (v_n, f_m^2)_{H^1}$. By construction, f_1^2 is close to f_1^1 . Notice further that $a_{21} = \|v_1\|_{H^1}^{-1} (v_2, v_1)_{H^1}$ can be made as small as necessary if θ_2 is sufficiently close to θ_1 . This implies that f_2^2 is close to f_2^1 . By induction we show that f_n^2 is close to f_n^1 for all $n = 1, 2, \dots, k$.

Defining $F_2 = \text{span}\{f_n^2 : n = 1, \dots, k\}$, we argue that F_2 satisfies the properties of the proposition. Indeed, by construction, $F_2 \subset V(\theta_2)$, $\dim F_2 = k$. Further, for fixed $u \in F_1$ one has $u = \sum_{n=1}^k b_n f_n^1$, for some $b_n \in \mathbb{C}$. The function $v = \sum_{n=1}^k b_n f_n^2$ belongs to F_2 and is close to u if θ_2 is sufficiently close to θ_1 . It follows that $\text{dist}(F_1, F_2) < \varepsilon$ when $|\theta_2 - \theta_1| < \delta$ for sufficiently small δ . Reversing the roles of u and v completes the proof. \square

THEOREM 5.2. *The eigenvalues $\lambda_k(\theta)$, $k \in \mathbb{N}$ are continuous functions of $\theta \in [0, 1)$.*

Proof. By a variational argument it is known that (see e.g. [15])

$$\lambda_k(\theta) = \inf_{\substack{F \subset V(\theta) \\ \dim F = k}} \sup_{u \in F} \frac{(A(\theta)u, u)}{\|u\|_{L^2(Q)}^2}, \quad (5.1)$$

Let $F_1 \subset V(\theta_1)$, $\dim F_1 = k$. For a fixed $\varepsilon > 0$, let δ , θ_2 and $F_2 \subset V(\theta_2)$ be given by Proposition 5.1. For a fixed $f_2 \in F_2$, there exists $f_1 \in F_1$ such that $\|f_1 - f_2\|_{H^1} \leq \varepsilon$.

This implies

$$\frac{(A(\theta_2)f_2, f_2)}{\|f_2\|_{L^2(Q)}^2} \leq \frac{(A(\theta_1)f_1, f_1)}{\|f_1\|_{L^2(Q)}^2} + \varepsilon \leq \sup_{u \in F_1} \frac{(A(\theta_1)u, u)}{\|u\|_{L^2(Q)}^2} + \varepsilon.$$

It follows, by the arbitrary choice of f_2 , that

$$\sup_{u \in F_2} \frac{(A(\theta_2)u, u)}{\|u\|_{L^2(Q)}^2} \leq \sup_{u \in F_1} \frac{(A(\theta_1)u, u)}{\|u\|_{L^2(Q)}^2} + \varepsilon.$$

Therefore, by the fact that F_1 is arbitrary and in view of the equation (5.1), we obtain

$$\begin{aligned} \inf_{F_2} \sup_{u \in F_2} \frac{(A(\theta_2)u, u)}{\|u\|_{L^2(Q)}^2} &\leq \inf_{\substack{F_1 \subset V(\theta_1) \\ \dim F_1 = k}} \sup_{u \in F_1} \frac{(A(\theta_1)u, u)}{\|u\|_{L^2(Q)}^2} + \varepsilon \\ &= \lambda_k(\theta) + \varepsilon. \end{aligned} \quad (5.2)$$

The set $\{F_2\}$ is a subset of the set of all k dimensional subspaces of $V(\theta_2)$, therefore

$$\lambda_k(\theta_2) = \inf_{\substack{F \subset V(\theta_2) \\ \dim F = k}} \sup_{u \in F} \frac{(A(\theta_2)u, u)}{\|u\|_{L^2(Q)}^2} \leq \inf_{F_2} \sup_{u \in F_2} \frac{(A(\theta_2)u, u)}{\|u\|_{L^2(Q)}^2}. \quad (5.3)$$

Equations (5.2) and (5.3) imply

$$\lambda_k(\theta_2) - \lambda_k(\theta_1) \leq \varepsilon.$$

Reversing the roles of θ_1 and θ_2 yields the desired result. \square

The above statement of continuity of $\lambda_k = \lambda_k(\theta)$, $k \in \mathbb{N}$, is not a simple consequence of the continuity of eigenvalues for the usual Floquet-Bloch decomposition. An important distinct feature of the present statement is the dependence on θ of the operator domain $V(\theta)$ in the variational principle (5.1).

REFERENCES

- [1] Allaire, G., 1992. Homogenization and two-scale convergence, *SIAM J. Math. Anal.* **23**, 1482–1518.
- [2] Allaire, G., Conca C. 1998. Bloch wave homogenization and spectral asymptotic analysis, *J. Math. Pures. Appl.* **77**, 153–208.
- [3] Bensoussan, A., Lions, J.-L., and Papanicolaou, G., 1978. *Asymptotic Analysis for Periodic Structures*, North-Holland.
- [4] Chandler-Wilde, S., Lindner, M., 2011. *Limit Operators, Collective Compactness, and the Spectral Theory of Infinite Matrices*, American Mathematical Society.
- [5] Cooper, S. 2012. Two-scale homogenisation of partially degenerating PDEs with applications to photonic crystals and elasticity, *PhD Thesis*, University of Bath.
- [6] Duvaut, G., Lions, J.-L., 1972. *Les Inéquations en Mécanique et en Physique*, Dunod, Paris.
- [7] Friesecke, G., James, R.D., Müller, S., 2002. A theorem on geometric rigidity and the derivation of nonlinear plate theory from three dimensional elasticity. *Comm. Pure Appl. Math.* **55** 1461–1506.
- [8] Fonseca, I., Leoni, G., Müller, S., 2004. \mathcal{A} -quasiconvexity: weak convergence and the gap, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **21**(2), 209–236.
- [9] Jackson, J. D., 1998. *Classical Electrodynamics*, John Wiley & Sons.
- [10] Kamotski, I. V., Smyshlyaev, V. P., 2011. Homogenisation of degenerate PDEs and applications to localisation of waves, *Preprint*, University College London.
- [11] Kato, T., 1995. *Perturbation Theory for Linear Operators*, Springer.

- [12] Murat, F., 1978. Compacité par compensation, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **5**, 489–507.
- [13] Nguetseng, G., 1989. A general convergence result for a functional related to the theory of homogenisation, *SIAM J. Math. Anal.* **20**, 608–623.
- [14] Ramakrishna, S. A., Grzegorzcyk, T.M., 2008. *Physics and Applications of Negative Refractive Index Materials*, CRC Press and SPIE Press.
- [15] Reed, M., Simon, B., 1978. *Methods of Modern Mathematical Physics IV: Analysis of Operators*, Academic Press.
- [16] Smyshlyaev, V. P., 2009. Propagation and localisation of elastic waves in highly anisotropic periodic composites via two-scale homogenisation. *Mechanics of Materials* **41** (2009), 434–447.
- [17] Zhikov, V. V., 2000. On an extension of the method of two-scale convergence and its applications, *Sb. Math.*, **191**(7), 973–1014.
- [18] Zhikov, V. V., 2002. Homogenisation of elasticity problems on singular structures, *Izvestiya RAN, Ser. Math.* **66** (2), 81–148.
- [19] Zhikov, V. V., 2005. On spectrum gaps of some divergent elliptic operators with periodic coefficients. *St. Petersburg Math. J.* **16**(5) (2005), 774–790.
- [20] Zolla, F., Renversez, G., Nicolet, A., Kuhlmei, B., Guenneau, S., Felbacq, D., Argyros, A., Leon-Saval, S., 2012. *Foundations of Photonic Crystal Fibres*, Imperial College Press.